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12.—Spectral Theory of Rotating Chains. By C. A. Stuart, Battelle Institute, 7 Route de Drize, Carouge, Geneva. *Communicated by* Dr J. B. McLeod

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Synopsis

Two eigenvalue problems associated with steady rotations of a chain are considered. To compare the spectra of these two problems, let $\sigma_K(n)$ denote the set of all angular velocities with which a chain of unit length with one end fixed and the other free can rotate in a vertical plane so as to have exactly n nodes on the vertical axis (including the fixed end). In a linearised theory $\sigma_K(n)$ is a single point, i.e.

$$\sigma_K(n) = \{\omega_n\} \quad \text{(linearised theory).}$$

In the full non-linear theory $\sigma_K(n)$ is an infinite interval lying to the *right* of ω_n. Indeed,

$$\sigma_K(n) = (\omega_n, \infty) \quad \text{(non-linear theory).}$$

This is established in [**1**].

Next, let $\sigma_M(\beta, n)$ denote the set of all angular velocities with which a chain, having ends fixed at unit distance apart on the vertical axis, can rotate in a vertical plane so as to have exactly $n+1$ nodes on the vertical axis (including the ends) and so that the tension takes the value β at the lower end. This problem, in which the length of the chain is not prescribed is a model for a spinning process in which the 'chain' is continuously created in a rotating configuration. For $\beta > 0$, we again have in the linearised theory that $\sigma_M(\beta, n)$ is a singleton, i.e.

$$\sigma_M(\beta, n) = \{\lambda_n(\beta)\} \quad \text{(linearised theory).}$$

In the full non-linear theory $\sigma_M(\beta, n)$ lies to the *left* of $\lambda_n(\beta)$. Although unable to determine exactly $\sigma_M(\beta, n)$ for $\beta > 0$, we have

$$(\lambda_n^*(\beta), \lambda_n(\beta)) \subset \sigma_M(\beta, n) \subset (\lambda_n'(\beta), \lambda_n(\beta)) \quad \text{(non-linear theory),}$$

where ω_n, $\lambda_n(\beta)$, $\lambda_n^*(\beta)$ and $\lambda_n'(\beta)$ are all characterised as the nth zeros of known combinations of Bessel functions.

1. Introduction

In his definitive article [**1**], Kolodner discussed the non-linear theory of a chain rotating steadily about the vertical axis through a fixed end whilst the other end is free and where at any instant the chain lies in a plane. According to the linear theory for such a chain of given length, there are only certain angular velocities $\{\omega_n\}_{n=1}^{\infty}$ with which the chain can rotate steadily in a position displaced from the vertical axis of rotation. For all other angular velocities the chain must lie along this vertical axis. The eigenvelocities $\{\omega_n\}_{n=1}^{\infty}$ form an increasing sequence which is characterised as the discrete spectrum of a linear boundary value problem. Kolodner showed that according to the more accurate non-linear theory such a chain can rotate steadily in a position displaced from the vertical axis of rotation at any angular velocity greater than ω_1, the lowest eigenvelocity of the linearised theory. Indeed, for $\omega_n < \omega \leqq \omega_{n+1}$, there are exactly n distinct displaced nodes of rotation together with the vertical configuration.

Here we return to the non-linear theory of a rotating chain under different boundary conditions. Both ends of the chain are considered to be fixed a given distance apart on the vertical axis and the tension is supposed to assume a given value at the lower end. In this case, however, the length of the chain is not prescribed and is to be determined as a function of the solution.

This problem has been posed by Dr F. J. Milford, Battelle Memorial Institute, Columbus, Ohio, as a possible model for a spinning process. The freedom of the length arises because the rotating thread between two fixed points is formed by the convergence of diffuse fibres. The twist in the thread is induced by the forced rotation of the end at which the spun thread is then drawn off.

It is shown below that, for each fixed value $\alpha>0$ of the tension, there exists a sequence $\{(v_n'(\alpha), v_n(\alpha))\}$ of open intervals such that, for a velocity of rotation between $v_n'(\alpha)$ and $v_n(\alpha)$, there exists a displaced mode of rotation about the axis of rotation satisfying the above boundary conditions and having exactly $(n-1)$ interior nodes. The upper limits, $v_n(\alpha)$, form an increasing sequence which is again characterised as the discrete spectrum of a linear Sturm-Liouville problem. Indeed the $v_n(\alpha)$ are the zeros of a known combination of Bessel functions of the first and second kind of order zero. However, the problem is not asymptotically linear and so the lower limits, $v_n'(\alpha)$, are not characterised as the spectrum of a linear problem. None the less, we do give explicit upper and lower bounds for $v_n'(\alpha)$, again in terms of the zeros of appropriate combinations of Bessel functions of order zero. For small $\alpha>0$, there do not exist two displaced modes with the same angular velocity but having different numbers of interior nodes. As the velocity of rotation approaches $v_n(\alpha)$ from below the corresponding displaced modes converge to the vertical configuration.

These results are obtained in section 3 by taking as starting point the Banach space theorem of Rabinowitz [**4**]. Then *a priori* bounds on the length of the chain and its maximum displacement from the vertical configuration are derived and subsequently used in conjunction with comparison arguments to yield the above results. Section 3 ends with some general remarks about the properties of displaced modes. In particular, we note that the bifurcation for the chain with a free end leads to modes with angular velocities greater than the bifurcation value whereas in the case of two fixed ends it leads to modes with angular velocities lower than the bifurcation value. To this extent, the two boundary value problems for the rotating chain described above exhibit complimentary situations for non-linear Sturm-Liouville problems.

A more precise description of the two boundary value problems for a chain is given in section 2 together with some notation. It is then shown that these problems are equivalent to eigenvalue problems associated with non-linear second-order ordinary differential equations on the unit interval.

2. Steadily Rotating Chains

Before proceeding with the analysis of the problems outlined in the introduction, it is convenient to fix some standard notation to be used throughout.

The real line is denoted by $\mathscr{R}$ and $\mathscr{R}_+ = \{\lambda \geqq 0\}$. The real Banach space of all continuous real-valued functions, u, on a compact interval $[a, b]$ with

$$\| u \| = \max \{| u(x)|: a \leqq x \leqq b\}$$

is denoted by $C[a, b]$.

We denote by $C^1[a, b]$ the real Banach space of all continuously differentiable functions u on a bounded open interval (a, b) such that u and u' can be considered to lie in $C[a, b]$. (Prime denotes differentiation.) The norm in $C^1[a, b]$ is taken to be

$$\| u \|_1 = \| u \| + \| u' \| \quad \text{for } u \in C^1[a, b].$$

We use $\mathcal{N}$ to denote the set of positive integers and, for $n \in \mathcal{N}$, we set
$S_n = \{u \in C^1[0, 1]$: u has exactly $n-1$ zeros in $(0, 1)$ and all the zeros of u in $[0, 1]$ are simple$\}$.

Let us now describe the mathematical model we shall use for the problems described in the introduction.

By a *chain* we mean a perfectly flexible inextensible material which at any given time occupies a simple curve. The mass is assumed to be uniformly distributed along such a curve, and we denote the mass per unit length by a constant $\rho > 0$ which is independent of time as well as position. We consider a chain with ends labelled A and B respectively. The end A is attached to a fixed point and we introduce a rectangular coordinate system (x_1, x_2, x_3) with origin at A and such that the force on a body due to gravity acts along the positive x_3-axis.

Since the chain is assumed *inextensible* we may choose the arc-length, s, measured along the chain as an independent variable in terms of which to describe the configuration of the chain at any time t. Let $s = 0$ at A and let $(u_1(s, t), u_2(s, t), u_3(s, t))$ denote the position at time t of the point s along the chain. Since the end A is fixed at the origin of coordinates we have

$$u_i(0, t) = 0 \quad \text{for } i = 1, 2, 3 \text{ and } t \geqq 0 \tag{2.1}$$

whilst s being arc-length implies that

$$\sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial s}(s, t) \right)^2 = 1 \quad \text{for } 0 < s < s_B \text{ and } t \geqq 0, \tag{2.2}$$

provided that u is differentiable, where s_B is used to denote the length of the chain.

Since the chain is *perfectly flexible*, the only force which the chain can exert at s is a *tension* which acts parallel to the tangent to the chain at s. Let $T(s, t)$ denote the tension at the point s at time t.

Assuming $\underline{u}$ and T are sufficiently smooth, the equations of motion are then

$$\rho \frac{\partial^2 u_i}{\partial t^2}(s, t) = \delta_{i3} \rho g + \frac{\partial}{\partial s} \left\{ T(s, t) \frac{\partial u_i}{\partial s}(s, t) \right\} \quad \text{for } 0 < s < s_B,\ t > 0 \text{ and } i = 1, 2, 3, \tag{2.3}$$

where δ_{ij} is the Kronecker delta and g is the acceleration due to gravity.

We seek only motions in which the chain lies in vertical plane which rotates with constant angular velocity ω about the x_3-axis, the chain being stationary relative to this rotating plane. That is, we assume that T is independent of t and that there exist functions v and w of s such that

$$\underline{u}(s, t) = (v(s) \cos \omega t, v(s) \sin \omega t, w(s)).$$

The equations (2.1)–(2.3) then yield,

$$v(0) = w(0) = 0 \tag{2.4}$$

$$v'(s)^2 + w'(s)^2 = 1 \tag{2.5}$$

$$-\rho\omega^2 v(s) = (T(s)v'(s))' \tag{2.6}$$

$$-\rho g = (T(s)w'(s))' \tag{2.7}$$

for $0 < s < s_B$, where prime denotes differentiation with respect to s.

We now define a *steady rotation of a chain of density* $\rho > 0$ to be a quintuple (s_B, v, w, T, ω) where $s_B > 0$, $\omega \geqq 0$; $v, w, T \in C^1[0, s_B]$; v and w are twice continuously differentiable on $(0, s_B)$, and the equations (2.4)–(2.7) are satisfied. A steady rotation is called *vertical* if $v \equiv 0$ and non-vertical motions are said to be *displaced*.

PROBLEM K. *Given $\rho > 0$ and $H > 0$, find those ω for which there exist displaced steady rotations (H, v, w, T, ω) of a chain of density ρ satisfying the additional boundary condition*

$$T(H) = 0. \tag{2.8}$$

The Problem K has the interpretation that the chain has a given length H and that the end-point B is free. Note that, for each $\rho > 0$, $H > 0$ and $\omega \geqq 0$, there is a vertical steady rotation satisfying (2.8) which is obtained by setting $v \equiv 0$, $w(s) = s$ and $T(s) = \rho g(H - s)$ for $0 \leqq s \leqq H$.

PROBLEM M. *Given $\rho > 0$, $L > 0$ and $\alpha \geqq 0$, find those ω for which there exist displaced steady rotations (s_B, v, w, T, ω) of a chain of density ρ satisfying the additional boundary conditions*

$$v(s_B) = 0, \tag{2.9}$$

$$w(s_B) = L, \tag{2.10}$$

$$w'(s_B) > 0, \tag{2.11}$$

and

$$T(s_B) = \alpha. \tag{2.12}$$

The boundary conditions (2.9) and (2.10) may be interpreted as fixing the end B at a distance L vertically below A. The condition (2.12) amounts to prescribing the tension at the lower point B, whilst, as we shall see, (2.11) implies that B is the lowest point of the chain. Note, however, that the length of the chain s_B is not prescribed; but, clearly, (2.10) implies that $s_B \geqq L$. For any $\rho > 0$, $L > 0$, $\omega \geqq 0$ and $\alpha \geqq 0$ there is again a vertical steady rotation satisfying (2.9)–(2.12) given by $s_B = L$, $v \equiv 0$, $w(s) = s$ and $T(s) = \alpha + \rho g\,(L - s)$ for $0 \leqq s \leqq L$. As will be shown in section 3, $T(0) = \alpha + \rho g L$ for any steady rotation satisfying (2.9)–(2.12) and consequently the results of section 3 also apply to the problem in which (2.12) is replaced by $T(0) = \alpha + \rho g L$. That is, a tension larger than that required to support a chain of density ρ and length L is prescribed at the upper fixed end A whilst the tension at the lower fixed end B is left free.

In terms of the variable s, the Problem M constitutes a free boundary value problem for the equations (2.5)–(2.7) since the value of s at which the boundary conditions (2.9)–(2.12) are applied is not prescribed. This complication is resolved if we choose x_3 as the independent variable. To verify that this is possible is the first step in the following result.

LEMMA. *Suppose that* (s_B, v, w, T, ω) *is a steady rotation of a chain of density* $\rho>0$ *such that*

$$T(s_B) \geqq 0 \quad \textit{and } w'(s_B)>0.$$

Then

$$T(s)>0 \quad \textit{for } 0 \leqq s<s_B \tag{2.13}$$

and

$$w'(s)>0 \quad \textit{for } 0 \leqq s \leqq s_B. \tag{2.14}$$

In particular, $w(s_B)>0$ *and* w *is a homeomorphism of* $[0, s_B]$ *on to* $[0, w(s_B)]$. *Let* $w(s_B) = L$ *and, for* $0 \leqq z \leqq 1$, *set*

$$s(z) = L^{-1}w^{-1}(Lz) \tag{2.15}$$

$$x(z) = L^{-1}v(Ls(z)) \tag{2.16}$$

$$\tilde{T}(z) = (\rho g L)^{-1}T(Ls(z)). \tag{2.17}$$

Then $s, x, \tilde{T} \in C^1[0, 1]$,

$$x(0) = 0$$

$$s(z) = \int_0^z (1+x'(t)^2)^{\frac{1}{2}}dt, \tag{2.18}$$

$$-\{(1+x'(z)^2)^{-\frac{1}{2}}\tilde{T}(z)x'(z)\}' = \lambda x(z)\{1+x'(z)^2\}^{\frac{1}{2}}, \tag{2.19}$$

$$-\{(1+x'(z)^2)^{-\frac{1}{2}}\tilde{T}(z)\}' = (1+x'(z)^2)^{\frac{1}{2}} \tag{2.20}$$

for $0<z<1$, *where* $\lambda = g^{-1}\omega^2L$ *and prime denotes differentiation with respect to* z.

Proof. By hypothesis we have $T(s_B)\, w'(s_B) \leqq 0$, whilst

$$-(T(s)w'(s))' = \rho g>0 \quad \text{for } 0<s<s_B \text{ by (2.7).}$$

Thus $T(s)w'(s)>0$ for $0 \leqq s<s_B$. However, $w'(s_B)>0$ and so this immediately yields (2.13) and (2.14). The remainder of the proof is an exercise in the change of variables.

PROBLEM M′. *Given* $\beta \geqq 0$, *find* $(x, \lambda) \in C^1[0, 1]\times\mathscr{R}_+$ *such that* $x \not\equiv 0$,

$$-\left\{\left[\beta(1+x'(1)^2)^{-\frac{1}{2}}+\int_z^1 (1+x'(t)^2)^{\frac{1}{2}}dt\right]x'(z)\right\}' = \lambda x(z)(1+x'(z)^2)^{\frac{1}{2}} \quad \text{for } 0<z<1, \tag{2.21}$$

$$x(0) = x(1) = 0 \tag{2.22}$$

and

$$x'(1)<\infty. \tag{2.23}$$

Problems M and M′ are equivalent.

If (s_B, v, w, T, ω) is a solution of Problem M for $L>0$ and $\alpha \geqq 0$, then, defining x, $\tilde{T}$ and λ as in Lemma 2.1, it follows that (x, λ) is a solution of Problem M′ for $\beta = (\rho g L)^{-1}\alpha$. Note that

$$(1+x'(z)^2)^{\frac{1}{2}}w'(Lz) = 1 \quad \text{for all } z \in (0, 1)$$

and so (2.23) follows from (2.11). Integrating (2.20) from z to 1 yields

$$\tilde{T}(z) = (1+x'(z)^2)^{\frac{1}{2}}\left\{\beta(1+x'(1)^2)^{-\frac{1}{2}} + \int_z^1 (1+x'(t)^2)^{\frac{1}{2}}dt\right\} \quad \text{for } 0 \leqq z \leqq 1 \tag{2.24}$$

and thus we see that (2.19) gives (2.21). Conversely, if (x, λ) is a solution of Problem M′ for $\beta \geqq 0$, let s and $\tilde{T}$ be defined by (2.18) and (2.24) respectively. Choose $L>0$ and set

$$v(Ls(z)) = Lx(z)$$

$$w(Ls(z)) = Lz$$

$$T(Ls(z)) = \rho g L\tilde{T}(z)$$

for $0 \leqq z \leqq 1$. It is again easily checked that $(Ls(1), v, w, T, \omega)$ is a solution of Problem M for L and $\alpha = \rho g L\beta$ where $w^2 = L^{-1}\lambda g$.

As is shown in [**1**], it is also convenient to replace Problem K by an equivalent problem, although this time we retain s as the independent variable.

PROBLEM K′. *Find* $(u, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ *such that* $u \not\equiv 0$,

$$-u''(y) = \lambda u(y)\{u(y)^2 + y^2\}^{-\frac{1}{2}} \quad \textit{for } 0<y<1 \tag{2.25}$$

and

$$u(0) = u'(1) = 0. \tag{2.26}$$

Solutions of Problems K and K′ are related by

$$s(y) = H(1-y), \quad \lambda = g^{-1}H\omega^2,$$

$$T(s(y)) = \rho g H\{u(y)^2 + y^2\}^{\frac{1}{2}}$$

$$v(s(y)) = \rho g H \int_0^{s(y)} T(r)^{-1}u(H^{-1}(H-r))dr$$

$$w(s(y)) = \rho g \int_0^{s(y)} T(r)^{-1}(H-r)dr$$

$$\text{for } 0 \leqq y \leqq 1 \text{ and } 0 \leqq s \leqq H.$$

Note that $u'(y) = g^{-1}\omega^2 v(s(y))$ for $0<y<1$, and so zeros of v correspond to zeros of u'. However, it is easily seen that u has exactly one zero between any two successive zeros of u'. Consequently v has the same number of nodes as u.

Thus both problems can be regarded as particular cases of the recently developed theory of non-linear Sturm-Liouville problems [2–9]. To describe the spectrum of Problem K′, let

$$\sigma_K(n) = \{\lambda\text{: there exists a solution } (u, \lambda) \text{ of (2.25)–(2.26) such that } u \in S_n\}.$$

THEOREM (Kolodner). *For each* $n \in \mathscr{N}$, $\sigma_K(n) = (\frac{1}{2}\omega_n^2, \infty)$ *where* ω_n *is the* nth *zero of* J_0.

The corresponding, but incomplete, result for Problem M′ is given in section 3. The general approach adopted there also yields very easily the above result of Kolodner, as is shown in [**13**].

3. Results for Problem M′

We begin by establishing some properties of solutions of Problem M′. Apart from giving a qualitative description of solutions of Problem M′, these results are also useful in the analysis of the spectral theory of Problem M′ given below.

Motivated by (2.18) and (2.24), throughout this section we use the notation

$$s(x)(z) = \int_0^z (1+x'(t)^2)^{\frac{1}{2}} dt \tag{3.1}$$

and

$$\tilde{T}(x, \beta)(z) = (1+x'(z)^2)^{\frac{1}{2}} \left\{ \beta(1+x'(1)^2)^{-\frac{1}{2}} + \int_z^1 (1+x'(t)^2)^{\frac{1}{2}} dt \right\} \tag{3.2}$$

for $x \in C^1[0, 1]$, $\beta \geqq 0$ and $0 \leqq z \leqq 1$.

Lemma 1. *Suppose that* $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ *is a solution of Problem M′ for* $\beta \geqq 0$. *Then*

$$\tilde{T}(x, \beta)(z) = 1 - z + \beta - \tfrac{1}{2}\lambda x(z)^2 \tag{3.3}$$

and

$$s(x)(1) - s(x)(z) = -\beta(1+x'(1)^2)^{-\frac{1}{2}} + \left\{ \beta^2(1+x'(1)^2)^{-1} + 2\int_z^1 \tilde{T}(x, \beta)(t) dt \right\}^{\frac{1}{2}} \tag{3.4}$$

for $0 \leqq z \leqq 1$.

Proof. Multiplying (2.21) by $x'(z)\{1+x'(z)^2\}^{-\frac{1}{2}}$, we have

$$-\{(1+x'(z)^2)^{\frac{1}{2}}\}' \left\{ \beta(1+x'(1)^2)^{-\frac{1}{2}} + \int_z^1 (1+x'(t)^2)^{\frac{1}{2}} dt \right\} + x'(z)^2 = \lambda x(z)x'(z) \quad \text{for } 0<z<1. \tag{3.5}$$

Integrating (3.5) from $z = y$ to $z = 1$ yields (3.3). But (3.2) can be written as

$$-\{\beta(1+x'(1)^2)^{-\frac{1}{2}} + s(x)(1) - s(x)(z)\}^{2\prime} = 2\tilde{T}(x, \beta)(z) \quad \text{for } 0<z<1.$$

Integrating this from $z = y$ to $z = 1$ yields a quadratic equation for $s(x)(1) - s(x)(z)$. The only non-negative solution of this equation is given by (3.4).

Corollary 2. *Suppose that* $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ *is a solution of Problem M′ for* $\beta \geqq 0$.

If

$$\beta = 0, \quad x \equiv 0. \tag{3.6}$$

If

$$\beta > 0, \quad x \in \{0\} \bigcup_{n=1}^{\infty} S_n. \tag{3.7}$$

Proof. Putting $z = \beta = 0$ in (3.4), we have

$$1 \leqq \int_0^1 (1+x'(t)^2)^{\frac{1}{2}} dt = \left\{ 1 - \lambda \int_0^1 x(t)^2 dt \right\}^{\frac{1}{2}} \leqq 1,$$

from which (3.6) follows easily. Suppose now that $\beta>0$ and that $x \notin \bigcup_{n=1}^{\infty} S_n$. Then there is a $z_0 \in [0, 1]$ such that $x(z_0) = x'(z_0) = 0$. However, since $\beta>0$, (2.21) can easily be written in the form

$$-x''(z) = f(z, x(z), x'(z)), \tag{3.8}$$

where $f(z, \cdot, \cdot)$ is a Lipschitz continuous function for each $z \in [0, 1]$. Furthermore, $x \equiv 0$ is the unique solution of (3.8) satisfying $x(z_0) = x'(z_0) = 0$. This establishes (3.7).

COROLLARY 3. *Suppose that $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ is a solution of Problem M' for $\beta>0$ and that $x \not\equiv 0$. Then*

$$\lambda>0,$$

$$-x(z)x''(z)>0 \quad \textit{if } x'(z)=0 \tag{3.9}$$

and

$$x'(z)x''(z)>0 \quad \textit{if } x(z)=0. \tag{3.10}$$

Let $0<z_1<z_2<1$ and $0 \leqq y_1 \leqq y_2 \leqq 1$ be zeros of x' and x respectively. Then

$$x(z_1)^2<x(z_2)^2 \quad \textit{and } x'(y_1)^2<x'(y_2)^2. \tag{3.11}$$

Proof. Suppose that $\lambda = 0$. Then multiplying (2.21) by $x(z)$ and integrating from $z = 0$ to $z = 1$ gives

$$0 = \int_0^1 \left\{\beta(1+x'(1)^2)^{-\frac{1}{2}}+ \int_z^1 (1+x'(t)^2)^{\frac{1}{2}}dt\right\} x'(z)^2 dz$$

which implies that $x \equiv 0$. Hence $\lambda>0$.

For $\lambda>0$, (2.21) can be rewritten as

$$-\left\{\beta(1+x'(1)^2)^{-\frac{1}{2}}+\int_z^1 (1+x'(t)^2)^{\frac{1}{2}}dt\right\} x''(z)+(1+x'(z)^2)^{\frac{1}{2}}x'(z) = \lambda x(z)(1+x'(z)^2)^{\frac{1}{2}}$$

from which (3.9) and (3.10) are easily deduced. In particular, the zeros of x' are simple and are separated by the zeros of x. Hence

$$z_2-z_1< \int_{z_1}^{z_2} (1+x'(t)^2)^{\frac{1}{2}}dt.$$

But (3.3) implies that

$$\int_{z_1}^{z_2} (1+x'(t)^2)^{\frac{1}{2}}dt = -z_1-\tfrac{1}{2}\lambda x(z_1)^2+z_2+\tfrac{1}{2}\lambda x(z_2)^2.$$

Thus we have

$$0<\tfrac{1}{2}\lambda\{x(z_2)^2-x(z_1)^2\}$$

from which it follows that $x(z_1)^2<x(z_2)^2$.

Suppose now that $x'(y_1)^2 \geqq x'(y_2)^2$. Then

$$\begin{aligned} y_2-y_1< \int_{y_1}^{y_2} (1+x'(z)^2)^{\frac{1}{2}}dz &= (1+\beta-y_1)(1+x'(y_1)^2)^{-\frac{1}{2}} \\ &\quad -(1+\beta-y_2)(1+x'(y_2)^2)^{-\frac{1}{2}} \\ &\leqq (y_2-y_1)(1+x'(y_2)^2)^{-\frac{1}{2}} \\ &\leqq y_2-y_1. \end{aligned}$$

Hence we have that $x'(y_1)^2<x'(y_2)^2$.

These inequalities can be interpreted as follows. Let $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ be a solution of Problem M′ for $\beta > 0$ such that $\lambda > 0$ and $x \not\equiv 0$. Suppose that

$$x(z_1) = x(z_2) = 0$$

and that $x(z) > 0$ for $z_1 < z < z_2$. Then both x' and x'' have exactly one zero (y_1 and y_2 respectively) in $[z_1, z_2]$ and $z_1 < y_2 < y_1 < z_2$. In the interval $[z_1, y_2]$, x is a strictly convex and is then strictly concave in the interval $[y_2, z_2]$. Similar statements can be made in the case $x(z) < 0$ for $z_1 < z < z_2$.

It is clear that, if $s(x)(1) = 1$, $x \equiv 0$. We now give a quantitative statement of this.

LEMMA 4. *Suppose that $x \in S_n$ for some $n \in \mathscr{N}$ and that $x(0) = x(1) = 0$. Then*

$$\| x \| \leqq \tfrac{1}{2}\{s(x)(1)^2 - 1\}^{\frac{1}{2}}. \tag{3.12}$$

Proof. Let the zeros of x be denoted by

$$0 = z_1 < z_2 < \ldots < z_{n+1} = 1,$$

and set

$$p_i = z_{i+1} - z_i,$$

$$m_i = \max\{| x(z)| : z_i \leqq z \leqq z_{i+1}\},$$

$$q_i = s(x)(z_{i+1}) - s(x)(z_i),$$

for

$$1 \leqq i \leqq n.$$

Note that $p_i < q_i$ and suppose that $z_i < y_i < z_{i+1}$ with

$$| x(y_i)| = m_i \quad \text{for } 1 \leqq i \leqq n.$$

Then

$$s(x)(y_i) - s(x)(z_i) \geqq \{m_i^2 + (y_i - z_i)^2\}^{\frac{1}{2}}$$

and

$$s(x)(z_{i+1}) - s(x)(y_i) \geqq \{m_i^2 + (z_{i+1} - y_i)^2\}^{\frac{1}{2}}$$

for

$$1 \leqq i \leqq n.$$

Hence

$$q_i \geqq \{4m_i^2 + p_i^2\}^{\frac{1}{2}}. \tag{3.13}$$

However, an easy induction argument shows that

$$\sum_{i=1}^{n} (q_i - p_i)^2 \leqq \left(\sum_{i=1}^{n} q_i\right)^2 - \left(\sum_{i=1}^{n} p_i\right)^2 \tag{3.14}$$

provided that $q_i \geqq p_i \geqq 0$ for all $i = 1, 2, \ldots, n$.
Thus

$$4 \| x \|^2 \leqq \sum_{i=1}^{n} (q_i - p_i)^2 \text{ by (3.13)}$$

$$\leqq s(x)(1)^2 - 1 \text{ by (3.14)}$$

and the proof is complete.

COROLLARY 5. *Suppose that $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ is a solution of Problem M′ for $\beta \geqq 0$. Then*

$$\tilde{T}(x, \beta)(0) = 1 + \beta, \tag{3.15}$$

$$1 \leqq s(x)(1) \leqq (1 + 2\beta)^{\frac{1}{2}}, \tag{3.16}$$

and

$$\| x \| \leqq (\tfrac{1}{2}\beta)^{\frac{1}{2}}. \tag{3.17}$$

Proof. Putting $z = 0$ in (3.3) gives (3.15). But

$$1 \leqq s(x)(1) = -\beta(1+x'(1)^2)^{-\frac{1}{2}} + \left\{\beta^2(1+x'(1)^2)^{-1}+1+2\beta-\lambda \int_0^1 x(z)^2 dz\right\}^{\frac{1}{2}}$$

by (3.4) and consequently (3.16) follows.

Finally (3.17) follows from Corollary 2 and Lemma 4.

Apart from these estimates, Lemma 4 also provides a useful lower bound for $s(x)(1)$ in terms of $x'(1)$.

LEMMA 6. *Suppose that* $(x, \lambda) \in C^1[0, 1] \times \mathscr{R}_+$ *is a solution of Problem M′. Then*

$$s(x)(1) \geqq -r + \{1 + r^2 + 8\beta(4+\lambda)^{-1}\}^{\frac{1}{2}} \tag{3.18}$$

where

$$r = 4\beta(4+\lambda)^{-1}(1+x'(1)^2)^{-\frac{1}{2}}.$$

Proof. Substituting (3.12) in (3.3), we have

$$2\int_0^1 \tilde{T}(x,\beta)(t)dt \geqq 1 + 2\beta - \tfrac{1}{4}\lambda\{s(x)(1)^2 - 1\},$$

and consequently (3.4) yields (3.18).

With these estimates at hand, let us now proceed with the spectral theory for Problem M′. According to Corollary 2, we need only consider $\beta > 0$ and, for $\beta > 0$, there exists a Green's function g_β for the problem,

$$-((1-z+\beta)x'(z))' = h(z), \quad 0 < z < 1 \tag*{$(3.19)_\beta$}$$

$$x(0) = x(1) = 0. \tag{3.20}$$

Setting

$$G_\beta(h)(z) = \int_0^1 g_\beta(z, y)h(y)dy \quad \text{for } 0 \leqq z \leqq 1, \tag{3.21}$$

we have that G_β: $C[0, 1] \to C^1[0, 1]$ is a compact linear operator and that $G_\beta(h)$ is the unique solution of $(3.19)_\beta$, (3.20) for each $h \in C[0, 1]$. Also for $\beta > 0$, we set

$$F_\beta(x, \lambda)(z) = \tilde{T}(x, \beta)(z)^{-1}\{\lambda x(z) - x'(z)\}\{1 - z + \beta - \tilde{T}(x, \beta)(z)\}$$

and note that F_β: $C^1[0, 1] \times \mathscr{R} \to C[0, 1]$ is a bounded continuous mapping. Furthermore,

$$\| x \|_1^{-1} \| F_\beta(x, \lambda) \| \to 0 \text{ as } \| x \|_1 \to 0 \text{ uniformly for } \lambda \text{ in bounded intervals.}$$

Rearranging (2.21) then leads us to the equation

$$x = \lambda G_\beta(x) - G_\beta(F_\beta(x, \lambda)), \quad (x, \lambda) \in C^1[0, 1] \times \mathscr{R} \tag*{$(3.23)_\beta$}$$

for fixed $\beta > 0$.

For each $\lambda \in \mathscr{R}$, the element $(0, \lambda) \in C^1[0, 1] \times \mathscr{R}$ is a solution of $(3.23)_\beta$ and such solutions are called trivial.

We denote by $\mathscr{S}(\beta)$ the subset of $C^1[0, 1] \times \mathscr{R}$ consisting of all non-trivial solutions of $(3.23)_\beta$. Note that elements of $\mathscr{S}(\beta)$ are solutions of Problem M′ for $\beta > 0$ and vice versa. Hence we have that

$$\mathscr{S}(\beta) \subset \bigcup_{n=1}^{\infty} S_n \text{ by (3.7),}$$

and

$$(-u, \lambda) \in \mathcal{S}(\beta) \text{ whenever } (u, \lambda) \in \mathcal{S}(\beta).$$

By (3.22), the linearisation of $(3.23)_\beta$ about the trivial solution $(0, \lambda)$ is

$$x = \lambda G_\beta(x), \quad (x, \lambda) \in C^1[0, 1] \times \mathcal{R}$$

and this is equivalent to the regular Sturm-Liouville problem,

$$-((1-z+\beta)x'(z))' = \lambda x(z), \quad 0<z<1 \qquad (3.24)_\beta$$

$$x(0) = x(1) = 0, \qquad (3.25)$$

by the definition of G_β for $\beta > 0$.

The general solution of $(3.24)_\beta$ is

$$x(z) = AJ_0(2\lambda^{\frac{1}{2}}(1-z+\beta)^{\frac{1}{2}}) + BY_0(2\lambda^{\frac{1}{2}}(1-z+\beta)^{\frac{1}{2}})$$

provided that $\lambda > 0$, where J_0 and Y_0 are the Bessel functions of order zero of the first and second kind and A, B are arbitrary constants. Hence the eigenvalues, $\lambda_n(\beta)$, of $(3.24)_\beta$, (3.25) are given by the zeros of $\mathcal{J}(\ \ , \beta)\colon (0, \infty) \to \mathcal{R}$, where

$$\mathcal{J}(\lambda, \beta) \equiv J_0(2\lambda^{\frac{1}{2}}(1+\beta)^{\frac{1}{2}})Y_0(2\lambda^{\frac{1}{2}}\beta^{\frac{1}{2}}) - J_0(2\lambda^{\frac{1}{2}}\beta^{\frac{1}{2}})Y_0(2\lambda^{\frac{1}{2}}(1+\beta)^{\frac{1}{2}}). \qquad (3.26)$$

The zeros of such combinations of Bessel functions are tabulated in [**11**] for example. Note that λ_n is a monotone increasing function of β for each $n \in \mathcal{N}$. Furthermore,

$$\lambda_n(\beta) \to \infty \quad \text{as} \quad \beta \to \infty \quad \text{for each } n \in \mathcal{N}$$

and

$$\lambda_n(\beta) \to \omega_n \equiv (\tfrac{1}{2}\sigma_n)^2 \quad \text{as} \quad \beta \to 0 \quad \text{for each } n \in \mathcal{N},$$

where σ_n is the nth zero of J_0.

Hence, for each fixed $\beta > 0$, we have

$$0 < \omega_1 < \lambda_1(\beta) < \lambda_2(\beta) < \ldots$$

and the eigenfunction

$$J_0(2\lambda_n(\beta)^{\frac{1}{2}}(1-z+\beta)^{\frac{1}{2}})Y_0(2\lambda_n(\beta)^{\frac{1}{2}}\beta^{\frac{1}{2}}) - Y_0(2\lambda_n(\beta^{\frac{1}{2}})(1-z+\beta)^{\frac{1}{2}})J_0(2\lambda_n(\beta)^{\frac{1}{2}}\beta^{\frac{1}{2}})$$

corresponding to $\lambda_n(\beta)$ belongs to S_n.

Let $\mathcal{S}'(\beta) = \mathcal{S}(\beta) \cup \{(0, \lambda_n(\beta)) \in C^1[0, 1] \times \mathcal{R}\colon n \in \mathcal{N}\}$. Then, for each $\beta > 0$, $\mathcal{S}'(\beta)$ is a metric space with the topology induced from $C^1[0, 1] \times \mathcal{R}$ and we denote by $\mathcal{C}_n(\beta)$ the component of $\mathcal{S}'(\beta)$ containing $(0, \lambda_n(\beta))$.

THEOREM 7. *For each $\beta > 0$ and $n \in \mathcal{N}$,*

$$u \in S_n \text{ whenever } (u, \lambda) \in \mathcal{C}_n(\beta) \setminus \{(0, \lambda_n(\beta))\} \qquad (3.27)$$

and

$$\{\| u' \| + | \lambda | \colon (u, \lambda) \in \mathcal{C}_n(\beta)\} = [0, \infty). \qquad (3.28)$$

Proof. It follows from Theorem 1.2 of [**4**] that either $\mathcal{C}_n(\beta)$ is an unbounded subset of $C^1[0, 1] \times \mathcal{R}$ or $(0, \lambda_m(\beta)) \in \mathcal{C}_n(\beta)$ for some $m \neq n$. However, (3.27) is established exactly as in section 2 of [**4**] and consequently the second alternative cannot occur. Thus (3.28) is also established since $\| u \|_1 \leqq 2 \| u' \|$ for all $u \in C^1[0, 1]$ with $u(0) = 0$.

As indicated in the introduction, the spectrum of Problem M′ can now be investigated by using appropriate comparison results from the theory of linear second order ordinary differential equations.

LEMMA 8. *Suppose that* $(x, \lambda) \in \mathscr{S}(\beta)$ *and that* $x \in S_n$ *for some* $\beta > 0$ *and* $n \in \mathcal{N}$. *Then* $(1+2\beta)^{-\frac{1}{2}}\omega_n < \lambda < \lambda_n(\beta)$.

Proof. Consider the linear eigenvalue problem,

$$-((1+x'(z)^2)^{-\frac{1}{2}}(1-z+\beta-\tfrac{1}{2}\lambda x(z)^2)v'(z))' = \mu(1+x'(z)^2)^{\frac{1}{2}}v(z), \quad 0<z<1 \tag{3.29}$$

$$v(0) = v(1) = 0. \tag{3.30}$$

This is a regular Sturm-Liouville problem since

$$1-z+\beta-\tfrac{1}{2}\lambda x(z)^2 > \beta(1+x'(1)^2)^{-\frac{1}{2}} > 0 \quad \text{for } 0 \leqq z \leqq 1$$

by (3.3). Furthermore

$$1-z+\beta-\tfrac{1}{2}\lambda x(z)^2 \leqq 1-z+\beta$$

and

$$(1+x'(z)^2)^{\frac{1}{2}} \geqq 1 \quad \text{for } 0 \leqq z \leqq 1.$$

Consequently, Theorem 7 on page 411 of [**12**] yields $\mu_n < \lambda_n(\beta)$ for $n \in \mathcal{N}$, where μ_n denotes the nth eigenvalue of (3.29), (3.30). But $x \in S_n$ is a solution of (3.29), (3.30) and so must be a multiple of the nth eigenfunction. That is, $\lambda = \mu_n$ and so we have $\lambda < \lambda_n(\beta)$.

To obtain a lower bound for λ, we choose a new independent variable y defined by

$$y(z) = s(x)(z),$$

and consider the Sturm-Liouville problem,

$$-\{[\beta(1+x'(1)^2)^{-\frac{1}{2}}+s(x)(1)-y](1+x'(z(y))^2)^{\frac{1}{2}}v'(y)\}' = \mu v(y) \quad \text{for } 0<y<s(x)(1), \tag{3.31}$$

$$v(0) = v(s(x)(1)) = 0. \tag{3.32}$$

The function w defined by

$$w(y(z)) = x(z) \quad \text{for } 0 \leqq z \leqq 1$$

is an eigenfunction of (3.31), (3.32) which has exactly $(n-1)$ interior zeros and $\mu = \lambda$ is the corresponding eigenvalue. That is $\mu_n = \lambda$ where μ_n is now the nth eigenvalue of (3.31), (3.32). However,

$$[\beta(1+x'(1)^2)^{-\frac{1}{2}}+s(x)(1)-y](1+x'(z(y))^2)^{\frac{1}{2}} > s(x)(1)-y \quad \text{for } 0<y<s(x)(1)$$

and so, again by Theorem 7 on page 411 of [**12**], μ_n must be larger than the nth eigenvalue of

$$-\{(s(x)(1)-y)v'(y)\}' = \mu v(y) \quad \text{for } 0<y<s(x)(1)$$

$$v(0) = 0, \quad |v(s(x)(1))| < \infty.$$

That is $\mu_n > s(x)(1)^{-1}\omega_n$, where ω_n is defined as above. Recalling the estimate (3.16), we thus have

$$\lambda > (1+2\beta)^{-\frac{1}{2}}\omega_n$$

completing the proof.

This result immediately enables Theorem 7 to be sharpened.

COROLLARY 9. *For each* $\beta > 0$ *and* $n \in \mathcal{N}$, *we have*

$$\{\lambda\colon (x, \lambda) \in \mathscr{C}_n(\beta)\} \subset ((1+2\beta)^{-\frac{1}{2}}\omega_n, \lambda_n(\beta)] \tag{3.33}$$

and

$$\{\| x' \|: (x, \lambda) \in \mathscr{C}_n(\beta)\} = [0, \infty). \tag{3.34}$$

Having found an interval containing $\{\lambda: (x, \lambda) \in \mathscr{C}_n(\beta)\}$, we now seek an interval contained in $\{\lambda: (x, \lambda) \in \mathscr{C}_n(\beta)\}$. For $\beta > 0$ and $n \in \mathscr{N}$, let

$$\Sigma(\beta, n) = \{\lambda \geqq 0\text{: there exists a sequence } \{(x_m, \mu_m)\} \subset \mathscr{C}_n(\beta)$$

such that $\mu_m \to \lambda$ and $\| x_m' \| \to \infty$ as $m \to \infty\}$.

Since the equation (2.21) is not asymptotically linear the usual methods, **[10]** of analysing $\Sigma(\beta, n)$ fail. However, we do have the following results.

LEMMA 10. *For each $\beta > 0$ and $n \in \mathscr{N}$, $\Sigma(\beta, n) \neq \varnothing$. Suppose that $\lambda \in \Sigma(\beta, n)$ and that $\{(x_m, \mu_m)\} \subset \mathscr{C}_n(\beta)$ is such that $\mu_m \to \lambda$ and $\| x_m' \| \to \infty$ as $m \to \infty$.*

Then $| x_m'(1)| \to \infty$ as $m \to \infty$.

Furthermore there exists a continuously differentiable function v on $[0, 1)$ such that $x_m \to v$ in $C^1[0, b]$ for all $b \in (0, 1)$.

Proof. It follows from Corollary 9 that $\Sigma(\beta, n) \neq \varnothing$. We assume therefore that $\lambda \in \Sigma(\beta, n)$ and that there exist sequences $\{t_m\} \subset [0, 1]$ and $\{(x_m, \mu_m)\} \subset \mathscr{C}_n(\beta)$ such that $\mu_m \to \lambda$ and $| x_m'(t_m)| \to \infty$ as $m \to \infty$. Then recalling (3.3), we have that

$$(1 - t_m) < \beta(1 + x_m'(1)^2)^{-\frac{1}{2}} + \int_{t_m}^1 (1 + x_m'(z)^2)^{\frac{1}{2}} dz \leqq (1 - t_m + \beta)(1 + x_m'(t_m)^2)^{-\frac{1}{2}}$$

for all $m \in \mathscr{N}$.

Since $x_m'(t_m)^2 \to \infty$ as $m \to \infty$ by hypothesis, this immediately yields, $t_m \to 1$ and $| x_m'(1)| \to \infty$ as $m \to \infty$. But x_m satisfies the equation

$$x_m''(z) = (1 + x_m'(z)^2)^{\frac{1}{2}}(x_m'(z) - \mu_m x_m(z)) \left\{ \beta(1 + x_m'(1)^2)^{-\frac{1}{2}} + \int_z^1 (1 + x_m'(t)^2)^{\frac{1}{2}} dt \right\}^{-1}$$

for $0 < z < 1$ and so

$$| x_m''(z)| \leqq (1 + x_m'(z)^2)^{\frac{1}{2}} \, | x_m'(z) - \mu_m x_m(z)| (1 - z)^{-1} \tag{3.35}$$

for $0 < z < 1$.

Since we have just shown that $\{x_m'\}$ is uniformly bounded on any compact subset of $[0, 1)$, it follows from (3.35) that $\{x_m''\}$ is also uniformly bounded on any compact subset of $[0, 1)$. The existence and properties of the function v in the statement of lemma now follow easily from the Ascoli-Arzela theorem.

Some further properties of v which will be useful are given in the following lemma.

LEMMA 11. *For v as defined in Lemma 10, we have that*

$$| v(z)| \leqq (\tfrac{1}{2}\beta)^{\frac{1}{2}} \quad \textit{for } 0 \leqq z < 1 \tag{3.36}$$

and

$$\beta^2(1 + 2\beta)^{-1} \leqq v'(0)^2 \leqq \beta(2 + \beta). \tag{3.37}$$

Furthermore, v' is Lebesgue integrable on $(0, 1)$ and

$$\int_0^1 (1 + v'(z)^2)^{\frac{1}{2}} dz = \lim_{m \to \infty} s(x_m)(1) = (1 + \beta)(1 + v'(0)^2)^{-\frac{1}{2}}. \tag{3.38}$$

Proof. The estimate (3.36) follows immediately from (3.17). From the Lebesgue Dominated Convergence theorem it then follows that

$$\lim_{m \to \infty} 2 \int_z^1 \{1 - t + \beta - \tfrac{1}{2}\mu_m x_m(t)^2\} dt$$

exists for each $z \in [0, 1]$ and is equal to

$$h(z) \equiv 2\int_z^1 \{1-t+\beta-\tfrac{1}{2}\lambda v(t)^2\}dt \quad \text{for } 0 \leqq z \leqq 1.$$

Note that $h(z) \geqq (1-z)^2 > 0$ for all $z \in [0, 1)$.

Hence, from (3.3) and (3.4), it now follows that $\lim_{m\to\infty} \{s(x_m)(1)-s(x_m)(z)\}$ exists and is equal to $h(z)^{\frac{1}{2}}$ for each $z \in [0, 1]$. Since $s(x_m)(0) = 0$ for all $m \in \mathcal{N}$, in particular, we have that $\lim_{m\to\infty} s(x_m)(1) = h(0)^{\frac{1}{2}}$ and consequently

$$\lim_{m\to\infty} s(x_m)(z) = h(0)^{\frac{1}{2}} - h(z)^{\frac{1}{2}} \quad \text{for all } z \in [0, 1].$$

Returning to (3.3), we then have

$$(1+v'(z)^2)^{\frac{1}{2}} = -\tfrac{1}{2}h'(z)h(z)^{-\frac{1}{2}} \quad \text{for all } z \in [0, 1)$$

and so

$$\int_0^t (1+v'(z)^2)^{\frac{1}{2}}dz = -\int_0^t \{h(z)^{\frac{1}{2}}\}'dz = h(0)^{\frac{1}{2}} - h(t)^{\frac{1}{2}} \quad \text{for all } t \in [0, 1).$$

Thus we have established that

$$\int_0^1 (1+v'(z))^2 dz \text{ exists and equals } h(0)^{\frac{1}{2}} = \lim_{m\to\infty} s(x_m)(1).$$

Since $|v'(z)| \leqq (1+v'(z)^2)^{\frac{1}{2}}$ for all $z \in [0, 1)$, v' is also seen to be integrable on $(0, 1)$. Putting $z = 0$ in (3.3), we have

$$s(x_m)(1) = (1+\beta)\{1+x_m'(0)^2\}^{-\frac{1}{2}}$$

from which we conclude that

$$\lim_{m\to\infty} s(x_m)(1) = (1+\beta)\{1+v'(0)^2\}^{-\frac{1}{2}}.$$

Recalling (3.16), this also yields

$$(1+\beta)\{1+v'(0)^2\}^{-\frac{1}{2}} \leqq (1+2\beta)^{\frac{1}{2}}$$

and thus

$$v'(0)^2 \geqq \beta^2(1+2\beta)^{-1}.$$

However, we also have that $s(x_m)(1) \geqq 1$ for all $m \in \mathcal{N}$ and so

$$(1+\beta)\{1+v'(0)^2\}^{-\frac{1}{2}} \geqq 1$$

which yields

$$v'(0)^2 \leqq \beta(2+\beta).$$

The proof of the lemma is now complete.

We can now use comparison theorems to find an interval containing $\Sigma(\beta, n)$.

LEMMA 12. *For each $\beta > 0$ and $n \in \mathcal{N}$, we have*

$$\Sigma(\beta, n) \subset [(1+2\beta)^{-\frac{1}{2}}\omega_n, \lambda_n^*(\beta)] \tag{3.39}$$

where ω_n is as defined above and

$$\lambda_n^*(\beta) = \{1+8\beta(4+\lambda_n(\beta))^{-1}\}^{-\frac{1}{2}}\lambda_n(\beta\{1+8\beta(4+\lambda_n(\beta))^{-1}\}^{-\frac{1}{2}}).$$

Proof. Suppose that $\lambda \in \Sigma(\beta, n)$ and that $\{(x_m, \mu_m)\} \subset \mathscr{C}_n(\beta)$ is such that $\mu_m \to \lambda$ and $|x_m'(1)| \to \infty$ as $m \to \infty$. Then $\lambda \geqq (1+2\beta)^{-\frac{1}{2}}\omega_n$ by (3.33).

To obtain an upper bound for λ, we consider the Sturm-Liouville problem,

$$-\{[1-z(y)+\beta-\tfrac{1}{2}\mu_m x_m(z(y))^2]v'(y)\}' = vv(y) \quad \text{for } 0<y<h_m \tag{3.40}$$

$$v(0) = v(h_m) = 0, \tag{3.41}$$

where $y(z) = s(x_m)(z)$ and $h_m = s(x_m)(1)$. Then, as in Lemma 8 we see that the function w defined by

$$w(y(z)) = x_m(z) \quad \text{for } 0 \leqq z \leqq 1$$

is an eigenfunction of (3.40), (3.41) which has exactly $n-1$ interior zeros in $(0, h_m)$. That is $\mu_m = \nu_n$ where ν_n is the nth eigenvalue of (3.40), (3.41).

However,

$$1-z(y)+\beta-\tfrac{1}{2}\mu_m x_m(z)^2 \leqq 1-z(y)+\beta < \int_{z(y)}^{1} (1+x_m'(t)^2)^{\frac{1}{2}} dt+\beta$$

$$= h_m - y + \beta \quad \text{for } 0 \leqq y < h_m$$

and so Theorem 7 on page 411 of [**12**] implies that ν_n is less than the nth eigenvalue of

$$-\{(h_m - y+\beta)v'(y)\}' = vv(y), \quad 0<y<h_m,$$

$$v(0) = v(h_m) = 0.$$

Hence we find that

$$\mu_m < h_m^{-1}\lambda_n(\beta h_m^{-1})$$

where $\lambda_n(\beta h_m^{-1})$ is defined as before to be the nth zero of $\mathscr{J}(\quad, \beta h_m^{-1})\colon (0, \infty)\to\mathscr{R}$ where $\mathscr{J}\colon (0, \infty)\times(0, \infty)\to\mathscr{R}$ is defined by (3.26). Therefore,

$$\lambda \leqq \lim_{m\to\infty} h_m^{-1}\lambda_n(\beta h_m^{-1})$$

$$= h_\infty^{-1}\lambda_n(\beta h_\infty^{-1}), \tag{3.42}$$

where

$$h_\infty = \lim_{m\to\infty} h_m \text{ exists by (3.38).}$$

But

$$h_\infty \geqq \{1+8\beta(4+\lambda)^{-1}\}^{\frac{1}{2}} \text{ by (3.18)}$$

$$\geqq \{1+8\beta(4+\lambda_n(\beta))^{-1}\}^{\frac{1}{2}} \tag{3.43}$$

since $\lambda \leqq \lambda_n(\beta)$.

As already noted, $\lambda_n\colon (0, \infty)\to\mathscr{R}$ is a strictly monotone increasing function and so the estimate $\lambda \leqq \lambda_n^*(\beta)$ follows from (3.42) and (3.43).

In order to summarise what has now been established about the spectrum of Problem M′, for $\beta>0$ and $n\in\mathscr{N}$, let

$$\sigma_M(\beta, n) = \{\lambda \geqq 0\colon \text{there exists } (x, \lambda)\in\mathscr{S}(\beta) \text{ with } x\in S_n\}.$$

THEOREM 13. *For each $\beta>0$ and $n\in\mathscr{N}$,*

$$(\lambda_n^*(\beta), \lambda_n(\beta)) \subset \sigma_M(\beta, n) \subset ((1+2\beta)^{-\frac{1}{2}}\omega_n, \lambda_n(\beta)).$$

Proof. From Lemma 8, it follows that

$$\sigma_M(\beta, n) \subset ((1+2\beta)^{-\frac{1}{2}}\omega_n, \lambda_n(\beta)).$$

Since $\Sigma(\beta, n) \neq \varnothing$, (3.39) implies that

$$(\lambda_n^*(\beta), \lambda_n(\beta)) \subset \{\lambda\colon (x, \lambda)\in\mathscr{C}_n(\beta)\} \subset \sigma_M(\beta, n)\cup\{\lambda_n(\beta)\}.$$

It should be noted that $\lambda_n^*(\beta) < \lambda_n(\beta)$ and that $\lambda_n(\beta) - (1+2\beta)^{-\frac{1}{2}}\omega_n$ tends to zero as $\beta \to 0$. In the case $\beta = 0$, we have from (3.6) that $\sigma_M(0, n) = \varnothing$ for all $n \in \mathcal{N}$. For small enough $\beta > 0$, we have $(1+2\beta)^{-\frac{1}{2}}\omega_{n+1} > \lambda_n(\beta)$ for all $n \leqq N$, and therefore $\sigma_M(\beta, n) \cap \sigma_M(\beta, m) = \varnothing$ provided that $m \neq n$ and $m, n \leqq N$.

In conclusion, we reinterpret some of the above results to describe a few properties of solutions of Problem M. For any solution (s_B, v, w, T, ω) of Problem M for $L > 0$ and $\alpha > 0$, we have that:

the tension, $T(0)$, at A is $\alpha + \rho g L$;

the length of the chain, s_B, does not exceed $(\rho g L + 2\alpha)^{\frac{1}{2}} L^{\frac{1}{2}} (\rho g)^{-\frac{1}{2}}$;

the displacement from the vertical axis does not exceed $(\alpha L)^{\frac{1}{2}} (2\rho g)^{-\frac{1}{2}}$.

For any displaced solution, v and v' have at most a finite number of zeros in $[0, s_B]$. Let s_i and t_i denote the ith zeros of v and v' respectively. Then

$$s_i < t_i < s_{i+1},$$

$$v(t_i)^2 < v(t_{i+1})^2 \quad \text{and} \quad v'(s_i)^2 < v'(s_{i+1})^2.$$

In particular, the maximum displacement of v occurs at the lowest turning value of v.

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